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Eric Olivier, Alain Thomas

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Weak Gibbs property and systems of numeration

par Éric OLIVIER et Alain THOMAS

Résumé. Nous étudions les propriétés d'autosimilarité et la nature gibbsienne de certaines mesures definies sur l'espace produit $\Omega_r := \{0, 1, \dots, r-1\}^{\mathbb{N}}$. Cet espace peut être identifié à l'intervalle $[0, 1]$ au moyen de la numération en base r. Le dernier paragraphe concerne la convolution de Bernoulli en base $\beta = \frac{1+\sqrt{5}}{2}$, appelée mesure de Erdős, et son analogue en base $-\beta = -\frac{1+\sqrt{5}}{2}$, que nous étudions au moyen d'un système de numeration approprié.

Abstract. We study the selfsimilarity and the Gibbs properties of several measures defined on the product space $\Omega_r := \{0, 1, \dots, r-1\}^{\mathbb{N}}$. This space can be identified with the interval $[0, 1]$ by means of the numeration in base r. The last section is devoted to the Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$, called the Erdős measure, and its analogue in base $-\beta = -\frac{1+\sqrt{5}}{2}$, that we study by means of a suitable system of numeration.

Key-words: Weak Gibbs measures, Bernoulli convolutions, β-numeration, Ostrowski numeration, infinite products of matrices.

2000 Mathematics Subject Classification: 28A12, 11A55, 11A63, 11A67, 15A48.

1. Introduction

One calls the Bernoulli convolution associated with the base $\beta > 1$ and the parameter vector $\mathbf{p} = (p_0, \dots, p_{s-1})$, the infinite product of the Dirac measures $p_0\delta_{\frac{0}{\beta^n}} + \cdots + p_{s-1}\delta_{\frac{s-1}{\beta^n}}$ for $n \geq 1$ (see [5, 19, 12, 13]). In other words, it is the distribution measure of the random variable defined by

$$X(\omega) = \sum_{n\geq 1} \frac{\omega_n}{\beta^n},$$

when $\omega = (\omega_n)_{n\in\mathbb{N}}$ has a Bernoulli distribution such that, for any $n \in \mathbb{N}$,

$$P(\omega_n = 0) = p_0, \dots, P(\omega_n = s-1) = p_{s-1}.$$

The Bernoulli convolution associated with β and $\mathbf{p}$ is the unique measure μ with bounded support that satisfies the self-similarity relation ([17]):

$$\mu = \sum_{i=0}^{s-1} p_i \cdot \mu \circ S_i^{-1},$$

where the affine contractions $S_i : \mathbb{R} \to \mathbb{R}$ are defined by $S_i(x) := \frac{x+i}{\beta}$. The measure μ is purely singular with respect to the Lebesgue measure when $\mathbf{p}$ is uniform and β a Pisot number, that is, the conjuguates of β have modulus less than 1. The problem to know if μ has the weak Gibbs property in the sense of Yuri [21] is not simple; it is solved in case β is a multinacci number ([6, 13]), but more complicated for other Pisot numbers of degree at least 3 (for instance in [13, Example 2.4], computing the values of the Bernoulli convolution in case $\beta^3 = 3\beta^2 - 1$ requires matrices of order 8).

Section 2 recalls the definition of the weak Gibbs property, and its link with the notions of Bernoulli or Markov measure.

Section 3 is devoted to some results of Mukherjea, Nakassis and Ratti about products of i. i. d. random stochastic matrices, that we present in a slighty different way (Proposition 3.1). They have computed the density of the limit distribution, in case this distribution is the Bernoulli convolution in base $\beta = \sqrt[m]{r}$ with parameters $p_0 = \cdots = p_{r-1} = \frac{1}{r}$.

The framework is different in the sections 5 to 7; we define a measure on $\Omega_r := \{0, 1, \ldots, r-1\}^{\mathbb{N}}$ by giving its values on the cylinders of Ω_r, under the form of products of 2×2 matrices and vectors. Theorem 6.1 gives a condition for such a measure, to be related to a Bernoulli convolution, via the representation of the reals in the integral base r. Establishing the weak Gibbs property requires the convergence of the involved product of matrices and vectors in the projective space of dimension 2. It is proved in [6] that the uniform Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$ is weak Gibbs; here, Section 7 give analogue result in the base $-\beta = -\frac{1+\sqrt{5}}{2}$.

2. Weak Gibbs measures

One says that the probability measure μ on the product space $\Omega_r = \{0, 1, \ldots, r-1\}^{\mathbb{N}}$ has the weak Gibbs property if there exists a map $\phi : \Omega_r \to \mathbb{R}$, continuous for the product topology on Ω_r, such that

$$\text{(1)} \qquad \lim_{n\to\infty} \left(\frac{\mu[\omega_1 \ldots \omega_n]}{e^{\phi(\omega)} e^{\phi(\sigma\omega)} \cdots e^{\phi(\sigma^{n-1}\omega)}} \right)^{1/n} = 1 \quad \text{uniformly on } \omega \in \Omega_r$$

(where σ is the shift on Ω_r , and $[\omega_1 \ldots \omega_n]$ is the cylinder of order n around ω that is, the set of the $\omega' \in \Omega_r$ such that $\omega'_i = \omega_i$ for $1 \leq i \leq n$). If (1) holds, ϕ is called a potential of μ.

Equivalently, μ has the weak Gibbs property if and only if the measure of any cylinder $[\omega_1 \dots \omega_n]$ can be approached by a product in the following way: there exists a continuous map $\varphi : \Omega_r \to]0, +\infty[$ such that

$$\forall \varepsilon > 0 \ \ \exists N \in \mathbb{N} \ \ \forall n \geq N \ \ \forall \omega \in \Omega_r$$
$$(\varphi(\omega) - \varepsilon) \dots (\varphi(\sigma^{n-1}\omega) - \varepsilon) \leq \mu[\omega_1 \dots \omega_n] \leq (\varphi(\omega) + \varepsilon) \dots (\varphi(\sigma^{n-1}\omega) + \varepsilon).$$

In case μ is σ-invariant, the following theorem gives an equivalent definition (see [8], [20], [15]), which involves the map ϕ_μ defined as follows:

$$\phi_\mu(\omega) := \lim_{n\to\infty} \log \frac{\mu[\omega_1 \dots \omega_n]}{\mu[\omega_2 \dots \omega_n]} \tag{2}$$

at each point $\omega \in \Omega_r$ such that this limit exists.

Theorem 2.1. *Let μ be a σ-invariant probability measure on Ω_r , and $\phi : \Omega_r \to \mathbb{R}$ a continuous map. The following assertions are equivalent:*

(i) μ is a weak Gibbs measure of potential ϕ and, for any $\omega \in \Omega_r$, $\sum_{a=0}^{r-1} e^{\phi(a\omega)} = 1$;

(ii) $\phi_\mu(\omega)$ exists for any $\omega \in \Omega_r$, and $\phi_\mu = \phi$;

(iii) μ has entropy $h_\mu = -\mu(\phi)$ and, for any $\omega \in \Omega_r$, $\sum_{a=0}^{r-1} e^{\phi(a\omega)} = 1$.

This theorem can be used to prove that a σ-invariant probability measure has the weak Gibbs property, by using the implication $(ii) \Rightarrow (i)$. Now for any probability measure μ on Ω_r, not necessarily σ-invariant, the following implication is straightforward (see [13]):

Proposition 2.2. *If ϕ_μ is defined and continuous on Ω_r , then μ is a weak Gibbs measure of potential ϕ_μ.*

The two following examples show that the Bernoulli and the Markovian measures are weak Gibbs. The third is a counterexample: the potential of the weak Gibbs measure μ_3 is not ϕ_{μ_3}.

Example. If μ_1 is a Bernoulli measure with support Ω_r , then ϕ_{μ_1} is the continuous map such that $\phi_{\mu_1}(\omega) = \log \mu_1[\omega_1]$ for any $\omega \in \Omega_r$.

Example. If μ_2 is a Markov measure with support Ω_r , then ϕ_{μ_2} is the continuous map such that $\phi_{\mu_2}(\omega) = \log \dfrac{\mu_2[\omega_1\omega_2]}{\mu_2[\omega_2]}$ for any $\omega \in \Omega_r$.

Example. (see [12]) Let the probability measure μ_3 be defined on Ω_r by

$$\mu_3[\omega_1 \dots \omega_n] := \frac{1}{2 \cdot (2r)^n} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1' & 0 \\ 1 & 1 \end{pmatrix} \dots \begin{pmatrix} \omega_n' & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $\omega_i' = 1 + \frac{2\omega_i}{r-1}$. Then μ_3 is weak Gibbs of potential $\phi : \omega \mapsto \log \frac{\omega_1'}{2r}$, although ϕ_{μ_3} is discontinuous at any point ω such that the series $S_\omega := \sum_{n\geq 1} \frac{1}{\omega_1' \dots \omega_n'}$ converges:

$$\phi_{\mu_3}(\omega) = \begin{cases} \log \frac{1}{2r} + \log(1 + \frac{1}{S_\omega}) & \text{if } S_\omega < \infty \\ \log \frac{1}{2r} & \text{if } S_\omega = \infty. \end{cases}$$

The notion of weak Gibbs measure generalize the one of Gibbs measure (see for instance [1]). Let us generalize in the same way the notion of quasi-Bernoulli measure (see [3], [7]), and say that μ is weakly quasi-Bernoulli if it satisfies the following condition:

(3)

$$\lim_{n\to\infty} \left(\frac{\mu[\omega_1 \dots \omega_n]}{\mu[\omega_1 \dots \omega_i]\mu[\omega_{i+1} \dots \omega_n]} \right)^{1/n} = 1 \quad \text{uniformly on } \omega \in \Omega_r \text{ and } i \in \{1, \dots, n\}.$$

Then one has the following

Proposition 2.3. *If a probability measure on Ω_r has the weak Gibbs property, it satisfies (3).*

This proposition is straightforward, but can be used to prove that a probability measure do not have the weak Gibbs property:

Example. Let μ_4 be defined on Ω_2 by

$$\mu_4[\omega_1 \dots \omega_n] := \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} M_{\omega_1} \dots M_{\omega_n} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $M_0 = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $M_1 = \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix}$. It is not weak Gibbs because $\left(\frac{\mu_4[1^n 0^n]}{\mu_4[1^n]\mu_4[0^n]} \right)^{1/n}$ do not converge to 1.

One can ask if the converse of Proposition 2.3 true, or if the condition (3) imply that μ is weak Bernoulli in the sense of Bowen [2].

3. Products of stochastic matrices

We consider a finite set of stochastic 2×2 matrices, let $M_k = \begin{pmatrix} x_k & 1 - x_k \\ y_k & 1 - y_k \end{pmatrix}$ for $k = 0, 1, \dots, r-1$, where $x_k, y_k \in [0, 1]$. We suppose the M_k are different from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

A. Mukherjea and al. have studied in [11] and [10] the distribution of the random matrice $\Omega_r \ni \omega \mapsto M_{\omega_1} \dots M_{\omega_n}$ when the distribution of ω is Bernoulli with positive parameters $p_0, \dots p_{r-1}$. This distribution converges when $n \to \infty$, though the matrix $M_{\omega_1} \dots M_{\omega_n}$ itself do not converge (that is, its entries are – in much cases – divergent sequences). But we shall prove the convergence of the matrix $M_{\omega_n}\dots M_{\omega_1}$ (which has of course the same distribution as $M_{\omega_1}\dots M_{\omega_n}$ when the distribution of ω is Bernoulli).

Proposition 3.1. *The product matrix $P_n^\omega := M_{\omega_n}\dots M_{\omega_1}$ converges uniformly on $\omega\in\Omega_r$ to the matrix $\begin{pmatrix} x^\omega & 1-x^\omega \\ x^\omega & 1-x^\omega \end{pmatrix}$, where $x^\omega := \sum_{i=1}^\infty y_{\omega_i}\det P_{i-1}^\omega$ and – by convention – P_0^ω is the unit-matrix.*

Proof. Setting $x_n^\omega := y_n^\omega + \det P_n^\omega$ with $y_n^\omega := \sum_{i=1}^n y_{\omega_i}\det P_{i-1}^\omega$ one check easily by induction that

$$P_n^\omega = \begin{pmatrix} x_n^\omega & 1-x_n^\omega \\ y_n^\omega & 1-y_n^\omega \end{pmatrix}.$$

The uniform convergence of the sequences x_n^ω and y_n^ω is due to the fact that each matrix M_k has – from the hypotheses – a determinant less than 1 in absolute value.

□

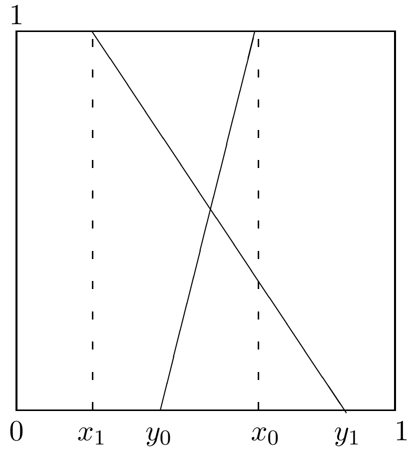
Theorem 3.2. *(*[11, Section 2]*) The distribution of $\omega\mapsto x^\omega$ is*
- *discrete if at least one of the matrices M_k is non invertible;*
- *singular continuous if the product $\left(\frac{|\det M_0|}{p_0}\right)^{p_0}\dots\left(\frac{|\det M_{r-1}|}{p_{r-1}}\right)^{p_{r-1}}$ belongs to $]0,1]$ and at least one of its factors is different from 1.*

Selfsimilarity relation: The random variable $\omega\mapsto x^\omega$ takes its values in $[0,1]$ because $\begin{pmatrix} x^\omega & 1-x^\omega \\ x^\omega & 1-x^\omega \end{pmatrix}$ is the limit of nonnegative matrices. Let λ be the probability distribution of $\omega\mapsto x^\omega$. If all the matrices M_k are invertible, then λ is selfsimilar in the sense that, for any borelian $B\subset[0,1]$,

$$\lambda(B) = \sum_{k=0}^{r-1} p_k\,\lambda\left(\frac{B-y_k}{x_k-y_k}\right)$$

(see [11, equation (2.6)] for the proof).

Let us represent, for instance in the case $r=2$ with $(x_0-y_0)(x_1-y_1)<0$, the two maps $x\mapsto\dfrac{x-y_k}{x_k-y_k}$ involved in the selfsimilarity relation:



Example. The probability distribution λ of $\omega \mapsto x^\omega$ is related to the numeration in a given base $\beta > 1$ if we suppose that $x_k = y_k + \frac{1}{\beta}$ and that $y_0, \dots, y_{r-1}$ are in arithmetic progression. Since we want that x_k and y_k belong to $[0,1]$, the good choice is

$$y_k = \frac{k}{r-1}\left(1 - \frac{1}{\beta}\right) \quad \text{for } k = 0, \dots, r-1.$$

Then $x_\omega = \frac{\beta-1}{r-1}\sum_{n\geq 1}\frac{\omega_n}{\beta^n}$ and $\lambda\left(\frac{\beta-1}{r-1}\cdot\right)$ is the convolution of the measures $p_0\delta_{\frac{0}{\beta^n}} + \cdots + p_{r-1}\delta_{\frac{r-1}{\beta^n}}$ for $n = 1, 2, \dots$.

In case $\beta = \sqrt[m]{r}$ with $m \in \mathbb{N}$, if the distribution is uniform ($p_0 = \cdots = p_{r-1} = \frac{1}{r}$), it is proved in [11, Proposition 1] that the density of the (absolutely continuous) distribution of $\omega \mapsto x^\omega$ is a piece-wise polynomial of degree at most m.

4. Uniform convergence (in direction) of the sequence of vectors $n \mapsto M_{\omega_1} \dots M_{\omega_n} V$

In this section $\mathcal{M} = \{M_0, \dots, M_{r-1}\}$ is a finite set of 2×2 matrices, where each matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}$ has nonnegative entries and each of the columns $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ is distinct from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. One denotes by
$\mathcal{M}^2$ the set of matrices MM' for M and M' in $\mathcal{M}$, and
$\mathcal{M}_1$ the set of matrices $M \in \mathcal{M}$ with $a = 0$;
$\mathcal{M}_2$ the set of matrices in $M \in \mathcal{M}^2$ with $b = 0$;
$\mathcal{M}_3$ the set of matrices in $M \in \mathcal{M}^2$ with $c = 0$;
$\mathcal{M}_4$ the set of matrices in $M \in \mathcal{M}$ with $d = 0$.

Proposition 4.1. *(*[12, theorem A]*) Let* $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *be a column matrix with positive entries. The sequence* $n \mapsto \dfrac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$ *converges uniformly on* $\omega \in \Omega_r$ *if and only if at least one of the following conditions holds:*

(i) $\nexists M \in \mathcal{M}_2$ *such that* $a > d$ *and* $\nexists M \in \mathcal{M}_3$ *such that* $a < d$ *and* $\mathcal{M}_2 \cap \mathcal{M}_3 = \emptyset$

(ii) $\nexists M \in \mathcal{M}_2$ *such that* $a \leq d$, *and* $\nexists M \in \mathcal{M}_3$ *such that* $a \geq d$

(iii) $\nexists M \in \mathcal{M}_2$ *such that* $a \leq d$ *and* $\nexists M \in \mathcal{M}_3$ *such that* $a < d$ *and* $\mathcal{M}_1 = \emptyset$

(iv) $\nexists M \in \mathcal{M}_2$ *such that* $a > d$ *and* $\nexists M \in \mathcal{M}_3$ *such that* $a \geq d$ *and* $\mathcal{M}_4 = \emptyset$

(v) V *is an eigenvector of all the matrices in* $\mathcal{M}$*.*

5. Application to the measures defined by products of matrices

Let $\mathcal{M} = \{M_0, \dots, M_{r-1}\}$ be a finite set of 2×2 matrices whose columns are distinct from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and let L (resp. V) be a positive row matrix (resp., a positive column matrix). If V is an eigenvector of $\sum_i M_i$ for the eigenvalue 1, one can define some measure η on Ω_r by setting

$$\eta[\omega_1 \dots \omega_n] = L M_{\omega_1} \dots M_{\omega_n} V.$$

Proposition 5.1. *The map* ϕ_η *defined in (2), exists and is continuous if and only if* $\mathcal{M}$ *satisfies at least one of the above conditions* (i), ..., (v), *or the following:*

(vi) L *is an eigenvector of all the matrices in* $\mathcal{M}$*.*

Proof. The map ϕ_η is related to the map $\psi_{\mathcal{M}} : \omega \mapsto \lim\limits_{n \to \infty} \dfrac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$. Indeed

$$\phi_\eta(\omega) = \frac{L M_{\omega_1} \psi_{\mathcal{M}} \circ \sigma(\omega)}{L \psi_{\mathcal{M}} \circ \sigma(\omega)}$$

for any $\omega \in \Omega_r$ such that $\psi_{\mathcal{M}} \circ \sigma(\omega)$ exists. Moreover if (vi) does not hold, the domains of definition of ϕ_η and $\psi_{\mathcal{M}} \circ \sigma$ are the same.

$\square$

Now this proposition gives a sufficient condition for η to have the weak Gibbs property (by using Proposition 2.2). This condition is of course not necessary (see Example 1.5).

6. Measures associated with the numeration in integral base r.

Let the map $X_{q,r} : \Omega_q \mapsto \left[0, \frac{q-1}{r-1}\right]$ be defined by

$$X_{q,r}(\omega) = \sum_{n\geq 1} \frac{\omega_n}{r^n}.$$

In particular $X_{r,r}$ is one-to-one except on a countable set because, if ω is not eventually $r-1$, the real $X_{r,r}(\omega)$ has expansion ω in base r. In the present section we identify the set of sequences Ω_r with the interval $[0,1]$, by means the map $X_{r,r}$.

The following theorem gives a condition for a measure defined by products of 2×2 matrices, to be related to some Bernoulli convolution in base r:

Theorem 6.1. *([12], Theorem 4.25) Let ν be a σ-invariant probability measure on Ω_r ; the following assertions are equivalent:*

(i) there exists a nonnegative row matrix L, a column matrix V and some square matrices $M_0, \dots, M_{r-1}$ such that

$$\forall \omega \in \Omega_r,\ \forall n \in \mathbb{N}, \quad \nu[\omega_1 \dots \omega_n] = L M_{\omega_1} \dots M_{\omega_n} V,$$

where the matrices $M_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ satisfy the conditions

$$b_0 = 0 \text{ and } \begin{pmatrix} a_k \\ c_k \end{pmatrix} = \begin{cases} \begin{pmatrix} b_{k+1} \\ d_{k+1} \end{pmatrix} & \text{if } 0 \leq k \leq r-2 \\ \begin{pmatrix} d_0 \\ b_0 \end{pmatrix} & \text{if } k = r-1 \end{cases}$$

(ii) there exists a nonnegative parameter vector $\mathbf{p} = (p_0, \dots, p_{2r-2})$ such that ν is the probability distribution $\nu_{\mathbf{p}}$ of the fractional part of $X_{2r-1,r}(\omega)$, when $\omega \in \Omega_{2r-1}$ has a Bernoulli distribution with parameter $\mathbf{p}$.

The relations between the matrices M_k and the parameter $\mathbf{p}$ are

$$p_0 = a_0, \dots, p_{r-1} = a_{r-1}, p_r = c_0, \dots, p_{2r-2} = c_{r-2}$$

and thus $\nu_{\mathbf{p}}$ is weak Gibbs from Proposition 5.1 in certain cases, for instance if the p_k are positive.

Selfimilarity relation Let $\mu_{\mathbf{p}}$ and $\nu_{\mathbf{p}}$ be the probability distributions of $X_{2r-1,r}$ and the fractionnal part of $X_{2r-1,r}$, respectively. Their respective supports are $[0,2]$ and $[0,1]$ and, for any borelian $B \subset [0,1]$,

$$\nu_{\mathbf{p}}(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B+1).$$

Theorem 6.1 is a consequence of the selfsimilarity relation

$$\mu_{\mathbf{p}}(B) = \sum_{k=0}^{2(r-1)} p_k \ \mu_{\mathbf{p}}(rB - k) \tag{4}$$

which allows to compute the column matrix $\begin{pmatrix} \mu_{\mathbf{p}}(B) \\ \mu_{\mathbf{p}}(B+1) \end{pmatrix}$.

The measure $\nu_{\mathbf{p}}$ has support $[0,1]$, while the measure $\nu_{\mathbf{p}}^*$ defined for any borelian $B \subset \mathbb{R}$, by

$$\nu_{\mathbf{p}}^*(B) = \mu_{\mathbf{p}}(B) + \mu_{\mathbf{p}}(B+1)$$

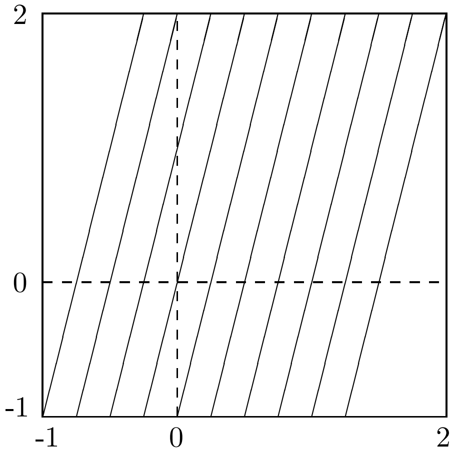
has support $[-1,2]$, and coincide with $\nu_{\mathbf{p}}$ on $[0,1]$. The selfsimilarity relation for $\nu_{\mathbf{p}}^*$ can be deduced from (4):

$$\nu_{\mathbf{p}}^*(B) = \sum_{k=-(r-1)}^{2(r-1)} p_k^* \, \nu_{\mathbf{p}}^*(rB - k) \tag{5}$$

where $p_k^* = \sum_{j \geq 0} (p_{k+2j} - p_{k+2j+1} + p_{k+2j+b} - p_{k+2j+b+1})$.

Both measures $\mu_{\mathbf{p}}$ and $\nu_{\mathbf{p}}^*$ are Bernoulli convolutions: they are – respectively – the infinite product of the measures $p_0 \delta_{\frac{0}{r^n}} + \cdots + p_{2r-2} \delta_{\frac{2r-2}{r^n}}$ and the one of the measures $p_{-r+1}^* \delta_{\frac{-r+1}{r^n}} + \cdots + p_{2r-2}^* \delta_{\frac{2r-2}{r^n}}$, for $n \geq 1$.

We represent below the maps $x \mapsto rx - k$ involved in (4) and (5), in the case $r = 4$:



7. The bases $\beta = \frac{1+\sqrt{5}}{2}$ and $-\beta = -\frac{1+\sqrt{5}}{2}$

We consider in this section the measures μ and $\mu_\star$ which are respectively the distributions of the random variables X and Y, defined by

$$X(\omega) = \sum_{n \geq 1} \frac{\omega_n}{\beta^{n+1}} \quad \text{and} \quad Y(\omega) = \frac{1}{\beta} - \sum_{n \geq 1} \frac{\omega_n}{(-\beta)^{n+1}},$$

when the distribution of $\omega \in \Omega_2$ is Bernoulli with positive parameter vector $\mathbf{p} = (p, q)$. We use consecutively two systems of numeration (see for instance [16], [14] and [4]): any real $x \in [0,1[$ can be represented in an unique way on the form

$$x = \sum_{n\geq 1} \frac{\varepsilon_n}{\beta^n} \quad \text{(Parry expansion)} \quad \text{and} \quad x = \frac{1}{\beta} - \sum_{n\geq 1} \frac{\alpha_n}{(-\beta)^{n+1}} \ ,$$

where $(\varepsilon_n)_{n\geq 1} =: \varepsilon(x)$ and $(\alpha_n)_{n\geq 1} =: \alpha(x)$ are two sequences with terms in $\{0,1\}$, without two consecutive terms 1, such that $\sigma^n\varepsilon(x)$ and $\sigma^{2n+1}\alpha(x)$ differ from the periodic sequence $1010\ldots$ for any $n \geq 0$. For any word $w = \omega_1 \ldots \omega_n$ on the alphabet $\{0,1\}$ and without factor 11, we denote

$$\begin{array}{rcl} [\![w]\!] & := & \{x \in [0,1[\ ;\ \varepsilon(x) \in [\omega_1,\ldots,\omega_n]\} \\ [\![w]\!]_\star & := & \{x \in [0,1[\ ;\ \alpha(x) \in [\omega_1,\ldots,\omega_n]\}\,. \end{array}$$

In case $\omega_n = 0$ we may compute $\mu[\![w]\!]$ and $\mu_\star[\![w]\!]_\star$ by the following formulas:

$$\text{(6)} \qquad \begin{pmatrix} \mu(\frac{1}{\beta}[\![w]\!]) \\ \mu(\frac{1}{\beta} + \frac{1}{\beta}[\![w]\!]) \\ \mu(\frac{1}{\beta^2} + \frac{1}{\beta}[\![w]\!]) \end{pmatrix} = M_{\omega_1} \ldots M_{\omega_n} \begin{pmatrix} \frac{p}{p+q^2} \\ \frac{q^2}{p+q^2} \\ \frac{q}{p+q^2} \end{pmatrix}$$

$$\text{(7)} \qquad \begin{pmatrix} \mu_\star([\![w]\!]_\star) \\ \mu_\star(-\frac{1}{\beta} + [\![w]\!]_\star) \\ \mu_\star(\frac{1}{\beta^2} + [\![w]\!]_\star) \end{pmatrix} = A_{\omega_1} \ldots A_{\omega_n} \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}$$

where

$$M_0 = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & q \\ q & p & 0 \end{pmatrix} \ M_1 = \begin{pmatrix} q & p & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix} \ A_0 = \begin{pmatrix} p & q & 0 \\ 0 & 0 & q \\ 0 & p & 0 \end{pmatrix} \ A_1 = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ p & q & 0 \end{pmatrix}.$$

The formula (6) – and its extension to the multinacci bases – is proved in [13]. Let us sketch the proof of (7), which is equivalent to the following (assuming again that the word w do not have two consecutive letters 1 and ends by the letter 0):

(8)

$$\begin{pmatrix} \mu_\star([\![w]\!]_\star) \\ \mu_\star(-\frac{1}{\beta} + [\![w]\!]_\star) \\ \mu_\star(\frac{1}{\beta^2} + [\![w]\!]_\star) \end{pmatrix} = A_{\omega_1} \begin{pmatrix} \mu_\star([\![w']\!]_\star) \\ \mu_\star(-\frac{1}{\beta} + [\![w']\!]_\star) \\ \mu_\star(\frac{1}{\beta^2} + [\![w']\!]_\star) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mu([0,1[) \\ \mu(-\frac{1}{\beta} + [0,1[) \\ \mu(\frac{1}{\beta^2} + [0,1[) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}$$

where $w' = \omega_2 \ldots \omega_n$ for $n \geq 2$ and, by convention, if $n = 1$ the word w' is empty and $[\![w']\!]_\star = [0,1[$. We first compute $\mu_\star([\![w]\!]_\star)$: it is the probability of the event $Y(\xi) \in [\![w]\!]_\star$. This event is equivalent to $Y(\sigma\xi) \in \frac{\omega_1 - \xi_1}{\beta} + [\![w']\!]_\star$ hence

- in case $\omega_1 = 0$, it is also equivalent to

$$\left\{ \begin{array}{l} \xi_1 = 0 \\ Y(\sigma\xi) \in [\![w']\!]_\star \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \xi_1 = 1 \\ Y(\sigma\xi) \in -\frac{1}{\beta} + [\![w']\!]_\star \end{array} \right.$$

and this explain why the first row in A_0 is $\begin{pmatrix} p & q & 0 \end{pmatrix}$;

• in case $\omega_1 = 1$ we have necessarily $n \geq 2$ and $\omega_2 = 0$, and the event $Y(\sigma\xi) \in \frac{1-\xi_1}{\beta} + [\![w']\!]_\star$ can occur only if $\xi_1 = 1$ and $Y(\sigma\xi) \in [\![w']\!]_\star$; so the first row in A_1 is $\begin{pmatrix} q & 0 & 0 \end{pmatrix}$.

• We compute in the same way $\mu_\star(-\frac{1}{\beta} + [\![w]\!]_\star)$ and $\mu_\star(\frac{1}{\beta^2} + [\![w]\!]_\star)$ and we conclude that the first equality in (8) is true.

• The second equality in (8) can be deduced from the first, by making $n = 1$ and $\omega_1 = 0$.

7.1. Bernoulli convolution in base $\beta = \frac{1+\sqrt{5}}{2}$ ([5]). The Gibbs properties of μ have been studied in [13] in the following sense: let be the words

$$w(0) := 00, \quad w(1) = 010 \quad \text{and} \quad w(2) = 10; \tag{9}$$

then for any $x \in [0,1[$, there exists a unique sequence $\xi(x) = (\xi_n)_{n\geq 1} \in \Omega_3$ such that the Parry expansion $\varepsilon(x)$ belongs for any $n \geq 1$ to the cylinder $[w(\xi_1 \dots \xi_n)]$, where $w(\xi_1 \dots \xi_n)$ is the concatenation of the words $w(\xi_1), \dots, w(\xi_n)$. The measure $\mu \circ \xi^{-1}$ is weak Gibbs on Ω_3 if and only if $p = q$ (this case is studied more in details in [6]); nevertheless $\phi_{\mu\circ\xi^{-1}}(100\dots) = \infty$ in this case.

7.2. Bernoulli convolution in base $-\beta = -\frac{1+\sqrt{5}}{2}$. The measure $\mu_\star$ has better Gibbs properties than μ: let us consider now – for any $x \in [0,1[$ – the sequence $\xi_\star(x) = (\xi_n^\star)_{n\geq 1} \in \Omega_3$ such that $\alpha(x) \in [w(\xi_1^\star \dots \xi_n^\star)]_\star$ for all $n \geq 1$, we have the following

Theorem 7.1. *(i) If $p \geq q$ the measure $\mu_\star \circ \xi_\star^{-1}$ is weak Gibbs on Ω_3 .*
(ii) if $p \leq q$ the measure $\mu_\star \circ S \circ \xi_\star^{-1}$ is weak Gibbs on Ω_3 , where $S(x) = 1 - x$ for any $x \in [0,1]$.

Proof. (ii) can be deduced from (i) by using the symmetry relation

$$Y(\omega_1\omega_2\dots) = 1 - Y((1-\omega_1)(1-\omega_2)\dots),$$

which implies $\mu_\star^{(p,q)} \circ S = \mu_\star^{(q,p)}$.

In order to prove (i), we don't use the matrices A_k but the product matrices associated to the three words defined in (9): setting $\alpha = \frac{p}{q}$ we have

$$A_0^* := {A_0}^2 = pq \begin{pmatrix} \alpha & 1 & \frac{1}{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1^* := A_0 A_1 A_0 = pq^2 \begin{pmatrix} \alpha & 1 & 0 \\ \alpha & 1 & \frac{1}{\alpha} \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2^* := A_1 A_0 = pq \begin{pmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 \\ \alpha & 1 & \frac{1}{\alpha} \end{pmatrix}.$$

Let us prove (i) by means of Proposition 2.2: more precisely we shall prove the uniform convergence of the (continuous) n-step potential $\phi_n : \Omega_3 \to \mathbb{R}$ defined by

(10)

$$\phi_n(\omega) := \log \frac{\mu_\star \circ \xi_\star^{-1}[\omega_1 \dots \omega_n]}{\mu_\star \circ \xi_\star^{-1}[\omega_2 \dots \omega_n]} = \log \frac{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A^*_{\omega_1} \dots A^*_{\omega_n} \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}}{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A^*_{\omega_2} \dots A^*_{\omega_n} \begin{pmatrix} 1 \\ \frac{q}{1+q} \\ \frac{1}{1+q} \end{pmatrix}}.$$

Notice that

(11)

$$A_0^{*n} = (pq)^n \begin{pmatrix} v_n(\alpha) & \alpha u_n(\alpha) & u_n(\alpha) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2^{*n} = (pq)^n \begin{pmatrix} 1 & 1/\alpha & 0 \\ 0 & 0 & 0 \\ u_n(1/\alpha) & u_n(1/\alpha)/\alpha & v_n(1/\alpha) \end{pmatrix}$$

where $u_n(x) := x^{-1} + x^0 + \cdots + x^{n-2}$ and $v_n(x) := x^n$ for any positive real x.

From now on we use the formalism of continued fractions ([18]) in a same way as in [13]: given n (odd) and $a_0 \geq 0$, $a_1 > 0, \dots, a_n > 0$ we put

$$\begin{pmatrix} p_{-1} \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ 1 \end{pmatrix} \quad \text{and, for } 1 \leq k \leq n, \begin{pmatrix} p_k \\ q_k \end{pmatrix} = u_k \begin{pmatrix} p_{k-1} \\ q_{k-1} \end{pmatrix} + v_{k-1} \begin{pmatrix} p_{k-2} \\ q_{k-2} \end{pmatrix}$$

where, for our purpose, $\begin{cases} u_i := u_{a_i}(\alpha) & (i \text{ even}) \\ u_i := u_{a_i}(1/\alpha) & (i \text{ odd}) \end{cases}$ and $\begin{cases} v_i := v_{a_i}(\alpha) & (i \text{ even}) \\ v_i := v_{a_i}(1/\alpha) & (i \text{ odd}). \end{cases}$

We have

$$\text{(12)} \qquad A_0^{*a_0} A_2^{*a_1} A_0^{*a_2} \dots A_2^{*a_n} = (pq)^{a_0 + \cdots + a_n} \begin{pmatrix} p_n & p_n/\alpha & v_n p_{n-1} \\ 0 & 0 & 0 \\ q_n & q_n/\alpha & v_n q_{n-1} \end{pmatrix}.$$

The difference $\delta_k = \left| \dfrac{p_k}{q_k} - \dfrac{p_{k-1}}{q_{k-1}} \right|$ is known to be at most $\dfrac{1}{a_1 + \cdots + a_k}$ in the case of the regular continued fractions ([9]) that is – with our notations – in the case $\alpha = 1$. We complete by the following

Lemma 7.2. *If $\alpha > 1$, then*

(i) for $k \geq 1$, $\delta_k \leq \dfrac{v_{k-1}}{u_k u_{k-1} + v_{k-1}} \delta_{k-1}$;

(ii) for $k \geq 1$ even, $\delta_k \leq \alpha^{1-(a_{k-1}+a_k)} \delta_{k-1}$;

(iii) for $k \geq 1$ even, $\delta_k \leq \alpha^{a_0 - (a_1 + \cdots + a_k)/2}$.

Proof. (i) By the definition of p_k and q_k ,

$$\begin{aligned}\frac{p_k}{q_k} &= \frac{u_k p_{k-1} + v_{k-1} p_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \\ &= \frac{u_k q_{k-1}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \frac{p_{k-1}}{q_{k-1}} + \frac{v_{k-1} q_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \frac{p_{k-2}}{q_{k-2}}\end{aligned}$$

hence

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{v_{k-1} q_{k-2}}{u_k q_{k-1} + v_{k-1} q_{k-2}} \cdot \left(\frac{p_{k-2}}{q_{k-2}} - \frac{p_{k-1}}{q_{k-1}} \right)$$

and, since $q_{k-1} \geq u_{k-1} q_{k-2}$, we are done.

(ii) If k is even one has $\dfrac{v_{k-1}}{u_k u_{k-1}} \leq \dfrac{\alpha^{-a_{k-1}}}{\alpha^{a_k - 2}\alpha}$, hence (i) implies (ii).

(iii) The inequalities (i) and (ii) imply respectively that the sequence (δ_k) is non-increasing and, if k is even, $\delta_k \leq \alpha^{-(a_{k-1}+a_k)/2}\delta_{k-1}$; hence $\delta_2 \leq \alpha^{-(a_1+a_2)/2}\delta_1$ and, by induction $\delta_k \leq \alpha^{-(a_1+\cdots+a_k)/2}\delta_1$ for any k even. Now $\delta_1 = \dfrac{v_0}{u_1} \leq \alpha^{a_0}$.

□

Notice that this lemma implies $\delta_k \leq \alpha^{a_0-(k-1)/2}$ for any $k \geq 1$, hence the sequence $k \mapsto \dfrac{p_k}{q_k}$ converges. Now we can prove the following

Lemma 7.3. *Suppose $\alpha \geq 1$ and let $\omega \in \Omega_3$.*

(i) At least one of the followings assertions is true:

$\exists N \geq 0$ such that $\omega_{N+1} \ldots \omega_{N+n} \in \{0,2\}^{n-1} \times \{2\}$ for infinitely many $n \geq 1$;

$\exists N \geq 0$ such that $\omega_{N+1} \ldots \omega_{N+n} \in \{0\}^n$ for all $n \geq 1$;

$\exists N \geq 0$ and $n \geq 2$ such that $\omega_{N+1} \ldots \omega_{N+n} \in \{1,2\} \times \{0\}^{n-2} \times \{1\}$.

(ii) In all cases there exists $N \geq 0$ and $n \geq 1$ such that

$$h, h' \geq N+n,\ \omega' \in [\omega_1 \ldots \omega_{N+n}] \Rightarrow \left|\phi_{h'}(\omega') - \phi_h(\omega)\right| \leq \varepsilon.$$

Proof. (i) If there exists $N \geq 0$ such that $\sigma^N \omega \in \{0,2\}^{\mathbb{N}}$, we are in the two first cases. If not, the digit 1 occurs infinitely many times in the sequence ω. The second occurrence of 1 is necessarily preceded by a word in $\{1,2\}\times\{0\}^k$ for some $k \geq 0$, hence we are in the third case.

(ii) Let N and n be as in (i). From (10), for any $h \geq N+n$ and $\omega' \in [\omega_1 \ldots \omega_{N+n}]$ there exists some reals $a, b, c, a', b', c', x, y, z$ such that

$$\phi_h(\omega') = \log \frac{\begin{pmatrix} a & b & c \end{pmatrix} A^*_{\omega_{N+1}} \ldots A^*_{\omega_{N+n}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\begin{pmatrix} a' & b' & c' \end{pmatrix} A^*_{\omega_{N+1}} \ldots A^*_{\omega_{N+n}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}, \tag{13}$$

where only x, y and z depend on h and ω'.

Suppose the first assertion in (i) is true. We deduce from the expression of $A^*_{\omega_{N+1}} \dots A^*_{\omega_{N+n}}$ in (12) that

$$\phi_h(\omega') = \log \frac{(ap_n + cq_n)(x + \frac{y}{\alpha}) + v_n(ap_{n-1} + cq_{n-1})z}{(a'p_n + c'q_n)(x + \frac{y}{\alpha}) + v_n(a'p_{n-1} + c'q_{n-1})z}.$$

This ratio lies between $\log \dfrac{ap_n + cq_n}{a'p_n + c'q_n}$ and $\log \dfrac{ap_{n-1} + cq_{n-1}}{a'p_{n-1} + c'q_{n-1}}$. These bounds do not depend on h nor ω', and converge – for $n \to \infty$ – to $\log \dfrac{a\theta + c}{a'\theta + c'}$ when $n \to \infty$, where $\theta := \lim\limits_{n\to\infty} \dfrac{p_n}{q_n}$. We deduce that (ii) is true in this case, by choosing n large enough.

The proof is similar when the second assertion in (i) is true, by using the expression of $A^*_{\omega_{N+1}} \dots A^*_{\omega_{N+n}}$ in (11).

If the third assertion in (i) is true, the matrix $A^*_{\omega_{N+1}} \dots A^*_{\omega_{N+n}}$ has rank 1; whence it maps $\mathbb{R}^3$ into a space of dimension 1, and the ratio in (13) do not depend on h nor ω' so that

$$h, h' \geq N + n,\ \omega' \in [\omega_1 \dots \omega_{N+n}] \Rightarrow \left|\phi_{h'}(\omega') - \phi_h(\omega)\right| = 0.$$

□

End of the proof of Theorem 7.1. Notice that Lemma 7.3(ii) implies – by making $\omega = \omega'$ – that the sequence $h \mapsto \phi_h(\omega)$ is Cauchy; let $\phi(\omega)$ be its limit.

Now we make $h, h' \to \infty$ in Lemma 7.3(ii): we obtain $|\phi(\omega') - \phi(\omega)| \leq \varepsilon$ for any ω' in the neighborhood $[\omega_1 \dots \omega_{N+n}]$ of ω, and this prove the continuity of ϕ so, by Proposition 2.2 $\mu_\star \circ \xi_\star^{-1}$ is weak Gibbs.

□

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Éric Olivier
Centre de Ressources Informatiques
Université de Provence
3, place Victor Hugo
13331 MARSEILLE Cedex 3, France
E-mail : `Eric.Olivier@up.univ-mrs.fr`

Alain Thomas
Centre de Mathématiques et d'Informatique, LATP Équipe de théorie des nombres
39, rue F. Joliot-Curie
13453 Marseille Cedex 13, France
E-mail : `thomas@cmi.univ-mrs.fr`